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Sequential product on standard effect algebra $\mathcal{E}(H)$

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Abstract
A quantum effect is an operator A on a complex Hilbert space H that satisfies $0 \leqslant A \leqslant I$, $\mathcal{E}(H)$ is the set of all quantum effects on H. In 2001, Professors Gudder and Nagy studied the sequential product $A \circ B = A^{\frac{1}{2}}BA^{\frac{1}{2}}$ for $A, B \in \mathcal{E}(H)$. In 2005, Professor Gudder asked: Is $A \circ B = A^{\frac{1}{2}}BA^{\frac{1}{2}}$ the only sequential product on $\mathcal{E}(H)$? Recently, Liu and Wu have presented an example to show that the answer is negative. In this paper, first, we characterize some algebraic properties of the abstract sequential product on $\mathcal{E}(H)$, second, we present a general method for constructing sequential products on $\mathcal{E}(H)$ and, finally, we study some properties of the sequential products constructed by the method.

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1. Introduction

The sequential effect algebra is an important model for studying quantum measurement theory [1–7]. A sequential effect algebra is an effect algebra which has a sequential product operation. First, we recall some elementary notations and results.

An *effect algebra* is a system $(E, 0, 1, \oplus)$, where 0 and 1 are distinct elements of E, and $\oplus$ is a partial binary operation on E satisfying that [8]

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(EA2) If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

(EA3) For each $a \in E$, there exists a unique element $b \in E$ such that $a \oplus b = 1$.

(EA4) If $a \oplus 1$ is defined, then $a = 0$.

In an effect algebra $(E, 0, 1, \oplus)$, if $a \oplus b$ is defined, we write $a \perp b$. For each $a \in (E, 0, 1, \oplus)$, it follows from (EA3) that there exists a unique element $b \in E$ such that $a \oplus b = 1$, we denote b by a'. Let $a, b \in (E, 0, 1, \oplus)$, if there exists $c \in E$ such that $a \perp c$ and $a \oplus c = b$, then we say that $a \leqslant b$. It follows from [8] that $\leqslant$ is a partial order of $(E, 0, 1, \oplus)$ and satisfies that for each $a \in E, 0 \leqslant a \leqslant 1, a \perp b$ if and only if $a \leqslant b'$.

Let $(E, 0, 1, \oplus, \circ)$ be an effect algebra and $a \in E$. If $a \wedge a' = 0$, then a is said to be a *sharp element* of E. We denote E_S by the set of all sharp elements of E [9, 10].

A *sequential effect algebra* is an effect algebra $(E, 0, 1, \oplus)$ with another binary operation $\circ$ defined on it satisfying [2]:

(SEA1) The map $b \mapsto a \circ b$ is additive for each $a \in E$, that is, if $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$.
(SEA2) $1 \circ a = a$ for each $a \in E$.
(SEA3) If $a \circ b = 0$, then $a \circ b = b \circ a$.
(SEA4) If $a \circ b = b \circ a$, then $a \circ b' = b' \circ a$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for each $c \in E$.
(SEA5) If $c \circ a = a \circ c$ and $c \circ b = b \circ c$, then $c \circ (a \circ b) = (a \circ b) \circ c$ and $c \circ (a \oplus b) = (a \oplus b) \circ c$ whenever $a \perp b$.

If $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra, then the operation $\circ$ is said to be a *sequential product* on $(E, 0, 1, \oplus)$. If $a, b \in (E, 0, 1, \oplus, \circ)$ and $a \circ b = b \circ a$, then a and b is said to be *sequentially independent* and is denoted by $a|b$ [1, 2].

Let H be a complex Hilbert space, $\mathcal{B}(H)$ be the set of all bounded linear operators on H, $\mathcal{P}(H)$ be the set of all projections on H, $\mathcal{E}(H)$ be the set of all self-adjoint operators on H satisfying that $0 \leqslant A \leqslant I$. For $A, B \in \mathcal{E}(H)$, we say that $A \oplus B$ is defined if $A + B \in \mathcal{E}(H)$; in this case, we define $A \oplus B = A + B$. It is easy to see that $(\mathcal{E}(H), 0, I, \oplus)$ is an effect algebra; we call it a *standard effect algebra* [8]. Each element A in $\mathcal{E}(H)$ is said to be a *quantum effect*; the set $\mathcal{E}(H)_S$ of all sharp elements of $(\mathcal{E}(H), 0, I, \oplus)$ is just $\mathcal{P}(H)$ [2, 9].

Let $A \in \mathcal{B}(H)$; we denote $\mathrm{Ker}(A) = \{x \in H \mid Ax = 0\}, \mathrm{Ran}(A) = \{Ax \mid x \in H\}$, $P_{\mathrm{Ker}(A)}$ denotes the projection onto $\mathrm{Ker}(A)$. Let $x \in H$ be a unit vector; P_x denotes the projection onto the one-dimensional subspace spanned by x.

In 2001 and 2002, Professors Gudder, Nagy and Greechie showed that for any two quantum effects A and B, if we define $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$, then the operation $\circ$ is a sequential product on the standard effect algebra $(\mathcal{E}(H), 0, I, \oplus)$; moreover, they studied some properties of this special sequential product on $(\mathcal{E}(H), 0, I, \oplus)$ [1, 2].

In 2005, Professor Gudder asked [4]: Is $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ the only sequential product on the standard effect algebra $(\mathcal{E}(H), 0, I, \oplus)$?

In 2009, Liu and Wu constructed a new sequential product on $(\mathcal{E}(H), 0, I, \oplus)$, thus answering Gudder's problem negatively [7]. This new sequential product on $(\mathcal{E}(H), 0, I, \oplus)$ motivated us to study the following topics in this paper:

(1) characterize the algebraic properties of an abstract sequential product on $(\mathcal{E}(H), 0, I, \oplus)$;
(2) present a general method for constructing a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$;
(3) characterize some elementary properties of the sequential product constructed by the method.

Our results generalize many conclusions in [1, 3, 7].

2. Abstract sequential product on $(\mathcal{E}(H), 0, I, \oplus)$

In this section, we study some elementary properties of the abstract sequential product on the standard effect algebra $(\mathcal{E}(H), 0, I, \oplus)$.

Lemma 2.1 [2]. *Let* $(E, 0, 1, \oplus, \circ)$ *be a sequential effect algebra,* $a \in E$. *Then the following conditions are all equivalent:*

(1) $a \in E_S$;
(2) $a \circ a' = 0$;
(3) $a \circ a = a$.

Lemma 2.2 [2]. *Let* $(E, 0, 1, \oplus, \circ)$ *be a sequential effect algebra,* $a \in E, b \in E_S$. *Then the following conditions are all equivalent:*

(1) $a \leqslant b$;
(2) $a \circ b = b \circ a = a$.

Lemma 2.3 [2, 8]. *Let* $(E, 0, 1, \oplus, \circ)$ *be a sequential effect algebra,* $a, b, c \in E$.

(1) If $a \perp b, a \perp c$ *and* $a \oplus b = a \oplus c$, *then* $b = c$.
(2) $a \circ b \leqslant a$.
(3) If $a \leqslant b$, *then* $c \circ a \leqslant c \circ b$.

Lemma 2.4 [7]. *Let* $\circ$ *be a sequential product on the standard effect algebra* $(\mathcal{E}(H), 0, I, \oplus)$. *Then for any* $A, B \in \mathcal{E}(H)$ *and real number* $t, 0 \leqslant t \leqslant 1$, *we have* $(tA) \circ B = A \circ (tB) = t(A \circ B)$.

Lemma 2.5 [1]. *Let* $A, B, C \in \mathcal{B}(H)$ *and* A, B, C *be self-adjoint operators. If for every unit vector* $x \in H$, $\langle Cx, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$, *then* $A = tI$ *or* $B = tI$ *for some real number t.*

Lemma 2.6 [11]. *Let* $A \in \mathcal{B}(H)$ *have the following operator matrix form:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with respect to the space decomposition $H = H_1 \oplus H_2$. *Then* $A \geqslant 0$ *iff*

(1) $A_{ii} \in \mathcal{B}(H_i)$ *and* $A_{ii} \geqslant 0, i = 1, 2$;
(2) $A_{21} = A_{12}^*$;
(3) there exists a linear operator D from H_2 *into* H_1 *such that* $\|D\| \leqslant 1$ *and* $A_{12} = A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}}$.

Theorem 2.1. *Let* $\circ$ *be a sequential product on* $(\mathcal{E}(H), 0, I, \oplus)$, $B \in \mathcal{E}(H)$, $E \in \mathcal{P}(H)$. *Then* $E \circ B = EBE$.

Proof. For $A \in \mathcal{E}(H)$, let $\Phi_A : \mathcal{E}(H) \longrightarrow \mathcal{E}(H)$ be defined by $\Phi_A(C) = A \circ C$ for each $C \in \mathcal{E}(H)$. It follows from lemma 2.4 and (SEA1) that Φ_A is affine on the convex set $\mathcal{E}(H)$. Note that $\mathcal{E}(H)$ generates algebraically the vector space $\mathcal{B}(H)$, so Φ_A has a unique linear extension to $\mathcal{B}(H)$, which we also denote by Φ_A. Then Φ_A is a positive linear operator on $\mathcal{B}(H)$ and $\Phi_A(I) = A$. Thus Φ_A is continuous.

Note that $E \in \mathcal{P}(H) = \mathcal{E}(H)_S$, it follows from lemma 2.1 that $E \circ (I - E) = 0$ and so $\Phi_E(I - E) = 0$. By composing Φ_E with all states on $\mathcal{B}(H)$ and using Schwarz's inequality, we conclude that $\Phi_E(B) = \Phi_E(EBE)$. Since $EBE \in \mathcal{E}(H)$, $E \in \mathcal{E}(H)_S$ and $EBE \leqslant E$, by lemma 2.2 we have $E \circ (EBE) = EBE$. Thus $E \circ B = \Phi_E(B) = \Phi_E(EBE) = E \circ (EBE) = EBE$.

In [7], the authors proved the above result for two-dimensional complex Hilbert spaces. □

Theorem 2.2. *Let* $\circ$ *be a sequential product on* $(\mathcal{E}(H), 0, I, \oplus)$, $A, B \in \mathcal{E}(H)$ *and* $AB = BA$. *Then* $A \circ B = B \circ A = AB$.

Proof. We use the notations as in the proof of theorem 2.1.

Suppose $E \in \mathcal{P}(H)$ and $E \in \{A\}'$, i.e., $EA = AE$. Note that EAE, $(I-E)A(I-E) \in \mathcal{E}(H)$, $EAE \leqslant E$ and $(I-E)A(I-E) \leqslant I-E$, by lemma 2.2, it follows that $EAE|E$ and $(I-E)A(I-E)|(I-E)$. Since $A = EAE + (I-E)A(I-E)$, by (SEA4) and (SEA5) we have $A|E$. By theorem 2.1, we conclude that $A \circ E = E \circ A = EAE = AE$. Thus, $\Phi_A(E) = AE$. Since Φ_A is a continuous linear operator and $\{A\}'$ is a von Neumann algebra, we conclude that $\Phi_A(B) = AB$. That is, $A \circ B = AB$. Similarly, we have $B \circ A = BA$. Thus $A \circ B = B \circ A = AB$. □

Theorem 2.3. *Let $\circ$ be a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$, $A, B \in \mathcal{E}(H)$. Then the following conditions are all equivalent:*

(1) $AB = BA = B$;
(2) $A \circ B \geqslant B$;
(3) $A \circ B = B$;
(4) $B \circ A = B$;
(5) $B \leqslant P_{\mathrm{Ker}(I-A)}$;
(6) $B \leqslant A^n$ *for each positive integer n.*

Proof. (1)⇒(3) and (1)⇒(4): by theorem 2.2.

(3)⇒(2) is obvious.

(4)⇒(3): by theorem 2.2, $B \circ A = B = B \circ I$. Thus, it follows from lemma 2.3 that $B \circ (I-A) = 0$. By (SEA3), $B|(I-A)$. By (SEA4), $B|A$. So $A \circ B = B \circ A = B$.

(2)⇒(6): by using theorem 2.2 and lemma 2.3 repeatedly, we have

$$\begin{aligned} &B \leqslant A \circ B \leqslant A \circ I = A;\\ &A \circ B \leqslant A \circ (A \circ B) \leqslant A \circ A = A^2;\\ &A \circ (A \circ B) \leqslant A \circ (A \circ (A \circ B)) \leqslant A \circ A^2 = A^3;\\ &\vdots\\ &A \circ \cdots \circ (A \circ B) \leqslant A \circ (A \circ \cdots \circ (A \circ B)) \leqslant A \circ A^{n-1} = A^n. \end{aligned}$$

The above showed that $B \leqslant A^n$ for each positive integer n.

(6)⇒(5): let $\chi_{\{1\}}$ be the characteristic function of $\{1\}$. Note that $0 \leqslant A \leqslant I$, it is easy to know that $\{A^n\}$ converges to $\chi_{\{1\}}(A) = P_{\mathrm{Ker}(I-A)}$ in the strong operator topology. Thus $B \leqslant P_{\mathrm{Ker}(I-A)}$.

(5)⇒(1): Since $0 \leqslant B \leqslant P_{\mathrm{Ker}(I-A)}$, we have $\mathrm{Ker}(P_{\mathrm{Ker}(I-A)}) \subseteq \mathrm{Ker}(B)$. So $\mathrm{Ran}(B) \subseteq \mathrm{Ran}(P_{\mathrm{Ker}(I-A)}) = \mathrm{Ker}(I-A)$. Thus $(I-A)B = 0$. That is, $AB = B$. Taking the adjoint, we get $AB = BA = B$. □

Theorem 2.4. *Let $\circ$ be a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$, $A, B \in \mathcal{E}(H)$. Then the following conditions are all equivalent:*

(1) $C \circ (A \circ B) = (C \circ A) \circ B$ *for every* $C \in \mathcal{E}(H)$;
(2) $\langle (A \circ B)x, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$ *for every* $x \in H$ *with* $\|x\| = 1$;
(3) $A = tI$ *or* $B = tI$ *for some real number* $0 \leqslant t \leqslant 1$.

Proof. By lemma 2.5, we conclude that (2)⇒(3). By theorem 2.2 and lemma 2.4, (3)⇒(1) is trivial.

(1)⇒(2): if (1) holds, then $P_x \circ (A \circ B) = (P_x \circ A) \circ B$ for every $x \in H$ with $\|x\| = 1$. By theorem 2.1, $P_x \circ (A \circ B) = P_x(A \circ B)P_x = \langle (A \circ B)x, x\rangle P_x$. By theorem 2.1 and lemma 2.4,

$(P_x \circ A) \circ B = (P_x A P_x) \circ B = (\langle Ax, x\rangle P_x) \circ B = \langle Ax, x\rangle(P_x \circ B) = \langle Ax, x\rangle P_x B P_x = \langle Ax, x\rangle\langle Bx, x\rangle P_x$. Thus (2) holds. □

Theorem 2.5. *Let* $\circ$ *be a sequential product on* $(\mathcal{E}(H), 0, I, \oplus)$, $B \in \mathcal{E}(H)$, $E \in \mathcal{P}(H)$. *Then the following conditions are all equivalent:*

(1) $E \circ B \leqslant B$;
(2) $EB = BE$;
(3) $E \circ B = B \circ E$.

Proof. (2)⇒(3): by theorem 2.2.

(3)⇒(1): by lemma 2.3.

(1)⇒(2): since $E \in \mathcal{P}(H)$, by theorem 2.1, $E \circ B = EBE$. Thus, $B - EBE \geqslant 0$. Note that

$$B - EBE = \begin{pmatrix} 0 & EB(I-E) \\ (I-E)BE & (I-E)B(I-E) \end{pmatrix}$$

with respect to the space decomposition $H = \overline{\mathrm{Ran}(E)} \oplus \mathrm{Ker}(E)$, so by lemma 2.6 we have $EB(I - E) = (I - E)BE = 0$. Thus $B = EBE + (I - E)B(I - E)$. So $EB = BE$. □

Theorem 2.6. *Let* $\circ$ *be a sequential product on* $(\mathcal{E}(H), 0, I, \oplus)$, $A, B, C \in \mathcal{E}(H)$. *If* A *is invertible, then the following conditions are all equivalent:*

(1) $B \leqslant C$;
(2) $A \circ B \leqslant A \circ C$.

Proof. (1)⇒(2): by lemma 2.3.

(2)⇒(1): it is easy to see that $\|A^{-1}\|^{-1}A^{-1} \in \mathcal{E}(H)$.

By lemma 2.3, $(\|A^{-1}\|^{-1}A^{-1}) \circ (A \circ B) \leqslant (\|A^{-1}\|^{-1}A^{-1}) \circ (A \circ C)$.

By theorem 2.2, $(\|A^{-1}\|^{-1}A^{-1})|A$ and $(\|A^{-1}\|^{-1}A^{-1}) \circ A = \|A^{-1}\|^{-1}I$.

By (SEA4) and theorem 2.2, we have

$$(\|A^{-1}\|^{-1}A^{-1}) \circ (A \circ B) = ((\|A^{-1}\|^{-1}A^{-1}) \circ A) \circ B = (\|A^{-1}\|^{-1}I) \circ B = \|A^{-1}\|^{-1}B,$$
$$(\|A^{-1}\|^{-1}A^{-1}) \circ (A \circ C) = ((\|A^{-1}\|^{-1}A^{-1}) \circ A) \circ C = (\|A^{-1}\|^{-1}I) \circ C = \|A^{-1}\|^{-1}C.$$

So, $B \leqslant C$. □

Corollary 2.1. *Let* $\circ$ *be a sequential product on* $(\mathcal{E}(H), 0, I, \oplus)$, $A, B, C \in \mathcal{E}(H)$. *If* A *is invertible, then the following conditions are all equivalent:*

(1) $B = C$;
(2) $A \circ B = A \circ C$.

3. General method for constructing sequential products on $\mathcal{E}(H)$

In the following, unless specified, let H be a finite-dimensional complex Hilbert space, **C** be the set of complex numbers, **R** be the set of real numbers, for each $A \in \mathcal{E}(H)$, $sp(A)$ be the spectrum of A and $\mathcal{B}(sp(A))$ be the set of all bounded complex Borel functions on $sp(A)$.

Let $A, B \in \mathcal{B}(H)$, if there exists a complex constant ξ such that $|\xi| = 1$ and $A = \xi B$, then we denote $A \approx B$.

In [7], Liu and Wu showed that if we define $A \circ B = A^{\frac{1}{2}} f_i(A) B f_{-i}(A) A^{\frac{1}{2}}$ for $A, B \in \mathcal{E}(H)$, where $f_z(t) = \exp z(\ln t)$ if $t \in (0, 1]$ and $f_z(0) = 0$, then $\circ$ is a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$; this result answered Gudder's problem negatively.

Now, we present a general method for constructing sequential products on $\mathcal{E}(H)$.

For each $A \in \mathcal{E}(H)$, take $f_A \in \mathcal{B}(sp(A))$.

Define $A \diamond B = f_A(A)B\overline{f_A}(A)$ for $A, B \in \mathcal{E}(H)$.

We say the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the *sequential product condition* if the following two conditions hold:

(i) for every $A \in \mathcal{E}(H)$ and $t \in sp(A)$, $|f_A(t)| = \sqrt{t}$;
(ii) for any $A, B \in \mathcal{E}(H)$, if $AB = BA$, then $f_A(A)f_B(B) \approx f_{AB}(AB)$.

If $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition, then it is easy to see that

(1) $f_A(A)\overline{f_A}(A) = \overline{f_A}(A)f_A(A) = A$, $(f_A(A))^* = \overline{f_A}(A)$;
(2) if $0 \in sp(A)$, then $f_A(0) = 0$;
(3) if $A = \sum_{k=1}^n \lambda_k E_k$, where $\{E_k\}_{k=1}^n$ are pairwise orthogonal projections, then $f_A(A) = \sum_{k=1}^n f_A(\lambda_k)E_k$;
(4) for each $E \in \mathcal{P}(H)$, $f_E(E) = f_E(0)(I - E) + f_E(1)E = f_E(1)E$;
(5) for any $A, B \in \mathcal{E}(H)$, $A \diamond B \in \mathcal{E}(H)$.

Lemma 3.1 [12]. *Let H be a complex Hilbert space, $A, B \in \mathcal{B}(H)$, A, B, AB be three normal operators, and at least one of A, B be a compact operator. Then BA is also a normal operator.*

Lemma 3.2 [13]. *If $M, N, T \in \mathcal{B}(H)$, M, N are normal operators and $MT = TN$, then $M^*T = TN^*$.*

Lemma 3.3. *Let $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfy the sequential product condition and $A, B \in \mathcal{E}(H)$. If $A \diamond B = B \diamond A$ or $A \diamond B = \overline{f_B}(B)Af_B(B)$, then $AB = BA$.*

Proof. If $A \diamond B = B \diamond A$, that is, $f_A(A)B\overline{f_A}(A) = f_B(B)A\overline{f_B}(B)$, then $f_A(A)\overline{f_B}(B)$ $f_B(B)\overline{f_A}(A) = f_B(B)\overline{f_A}(A)f_A(A)\overline{f_B}(B)$, so $f_A(A)\overline{f_B}(B)$ is normal. By lemma 3.1, we have that $\overline{f_B}(B)f_A(A)$ is also normal. Note that $(f_A(A)\overline{f_B}(B))f_A(A) = f_A(A)(\overline{f_B}(B)f_A(A))$, by using lemma 3.2, we have $(f_A(A)\overline{f_B}(B))^*f_A(A) = f_A(A)(\overline{f_B}(B)f_A(A))^*$. That is, $f_B(B)A = Af_B(B)$. Taking the adjoint, we have $\overline{f_B}(B)A = A\overline{f_B}(B)$. Thus, $AB = A\overline{f_B}(B)f_B(B) = \overline{f_B}(B)Af_B(B) = \overline{f_B}(B)f_B(B)A = BA$.

If $A \diamond B = \overline{f_B}(B)Af_B(B)$, that is, $f_A(A)B\overline{f_A}(A) = \overline{f_B}(B)Af_B(B)$, the proof is similar, we omit it. □

Lemma 3.4. *Let $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfy the sequential product condition and $A, B \in \mathcal{E}(H)$. If $AB = BA$, then $A \diamond B = B \diamond A = AB$.*

Proof. Since $AB = BA$, by sequential product condition (i) we have $A \diamond B = f_A(A)B\overline{f_A}(A) = |f_A|^2(A)B = AB$. Similarly, $B\diamond A = f_B(B)A\overline{f_B}(B) = |f_B|^2(B)A = AB$. Thus $A \diamond B = B \diamond A = AB$. □

Lemma 3.5. *Let $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfy the sequential product condition and $A, B \in \mathcal{E}(H)$. If $AB = BA$, then for every $C \in \mathcal{E}(H)$, $A \diamond (B \diamond C) = (A \diamond B) \diamond C$.*

Proof. By lemma 3.4, $A\diamond B = AB$. By sequential product condition (ii), there exists a complex constant ξ such that $|\xi| = 1$ and $f_A(A)f_B(B) = \xi f_{AB}(AB)$. Taking the adjoint, we have $\overline{f_B}(B)\overline{f_A}(A) = \overline{\xi}\, \overline{f_{AB}}(AB)$. Thus, $f_A(A)f_B(B)C\overline{f_B}(B)\overline{f_A}(A) = f_{AB}(AB)C\overline{f_{AB}}(AB) = f_{A\diamond B}(A \diamond B)C\overline{f_{A\diamond B}}(A \diamond B)$. That is, $A \diamond (B \diamond C) = (A \diamond B) \diamond C$. □

Lemma 3.6 [1]. *If $y, z \in H$ and $|\langle y, x\rangle| = |\langle z, x\rangle|$ for every $x \in H$, then there exists $c \in \mathbf{C}$, $|c| = 1$, such that $y = cz$.*

Lemma 3.7 [14]**.** *Let $f : H \longrightarrow \mathbf{C}$ be a mapping, $T \in \mathcal{B}(H)$. If the operator $S : H \longrightarrow H$ defined by $S(x) = f(x)T(x)$ is linear, then $f(x) = f(y)$ for every $x, y \notin \mathrm{Ker}(T)$.*

Lemma 3.8. *Let $f : H \longrightarrow \mathbf{C}$ be a mapping, $T \in \mathcal{B}(H)$. If the operator $S : H \longrightarrow H$ defined by $S(x) = f(x)T(x)$ is linear, then there exists a constant $\xi \in \mathbf{C}$ such that $S(x) = \xi T(x)$ for every $x \in H$.*

Proof. By lemma 3.7, there exists a constant $\xi \in \mathbf{C}$ such that $S(x) = \xi T(x)$ for every $x \notin \mathrm{Ker}(T)$. Of course, $S(x) = 0 = \xi T(x)$ for every $x \in \mathrm{Ker}(T)$. So $S(x) = \xi T(x)$ for every $x \in H$. □

Our main result in the section is the following.

Theorem 3.1. *For each $A \in \mathcal{E}(H)$, take $f_A \in \mathcal{B}(sp(A))$. Define $A \diamond B = f_A(A)B\overline{f_A}(A)$ for $B \in \mathcal{E}(H)$. Then $\diamond$ is a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$ iff the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition.*

Proof.

(1) First, we suppose that $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition, we show that $(\mathcal{E}(H), 0, I, \oplus, \diamond)$ is a sequential effect algebra.
(SEA1) is obvious.
By lemma 3.4, $I \diamond B = B$ for each $B \in \mathcal{E}(H)$, so (SEA2) hold.
We verify (SEA3) as follows: if $A \diamond B = 0$, then $f_A(A)B\overline{f_A}(A) = 0$, so $f_A(A)B^{\frac{1}{2}} = 0$, thus, we have $AB = \overline{f_A}(A)f_A(A)B^{\frac{1}{2}}B^{\frac{1}{2}} = 0$. Taking the adjoint, we have $AB = BA$. So $A \diamond B = B \diamond A$. We verify (SEA4) as follows: if $A \diamond B = B \diamond A$, then by lemma 3.3, $AB = BA$. So $A(I - B) = (I - B)A$. By lemma 3.4, we have $A \diamond (I - B) = (I - B) \diamond A$. By lemma 3.5, $A \diamond (B \diamond C) = (A \diamond B) \diamond C$ for every $C \in \mathcal{E}(H)$. We verify (SEA5) as follows: if $C \diamond A = A \diamond C$ and $C \diamond B = B \diamond C$, then by lemma 3.3, $AC = CA$, $BC = CB$. So (SEA5) follows easily by lemma 3.4. Thus, we proved that $(\mathcal{E}(H), 0, I, \oplus, \diamond)$ is a sequential effect algebra.

(2) Now we suppose that $\diamond$ is a sequential product on $(\mathcal{E}(H), 0, I, \oplus)$, we show that the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition. Since $(\mathcal{E}(H), 0, I, \oplus, \diamond)$ is a sequential effect algebra, by theorem 2.2, for each $A \in \mathcal{E}(H)$, $A \diamond I = A$, thus $|f_A|^2(A) = A$. If $A = \sum_{k=1}^{n} \lambda_k E_k$, where $\{E_k\}_{k=1}^{n}$ are pairwise orthogonal projections, $\sum_{k=1}^{n} E_k = I$, then $sp(A) = \{\lambda_k\}$, $|f_A|^2(A) = \sum_{k=1}^{n} |f_A(\lambda_k)|^2 E_k$. Thus $|f_A(\lambda_k)| = \sqrt{\lambda_k}$ and $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies sequential product condition (i).

To prove that $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies sequential product condition (ii), let $A, B \in \mathcal{E}(H)$ and $AB = BA$. By theorem 2.2, we have $A \diamond B = B \diamond A = AB$. Thus by (SEA4), $A \diamond (B \diamond C) = (A \diamond B) \diamond C$ for every $C \in \mathcal{E}(H)$.

Let $x \in H$, $\|x\| = 1$, $C = P_x$. Then for every $y \in H$, we have

$$\begin{aligned}\langle f_A(A)f_B(B)P_x\overline{f_B}(B)\overline{f_A}(A)y, y\rangle &= \langle (A \diamond (B \diamond P_x))y, y\rangle \\ &= \langle ((A \diamond B) \diamond P_x)y, y\rangle \\ &= \langle ((AB) \diamond P_x)y, y\rangle \\ &= \langle f_{AB}(AB)P_x\overline{f_{AB}}(AB)y, y\rangle.\end{aligned}$$

Since

$$\langle f_A(A)f_B(B)P_x\overline{f_B}(B)\overline{f_A}(A)y, y\rangle = |\langle \overline{f_B}(B)\overline{f_A}(A)y, x\rangle|^2,$$
$$\langle f_{AB}(AB)P_x\overline{f_{AB}}(AB)y, y\rangle = |\langle \overline{f_{AB}}(AB)y, x\rangle|^2,$$

we have $|\langle \overline{f_B}(B)\overline{f_A}(A)y, x\rangle| = |\langle \overline{f_{AB}}(AB)y, x\rangle|$ for every $x, y \in H$.

By lemma 3.6, there exists a complex function g on H such that $|g(x)| \equiv 1$ and $\overline{f_B}(B)\overline{f_A}(A)x = g(x)\overline{f_{AB}}(AB)x$ for every $x \in H$. By lemma 3.8, there exists a constant $\xi \in \mathbf{C}$ such that $|\xi| = 1$ and $\overline{f_B}(B)\overline{f_A}(A)x = \xi\overline{f_{AB}}(AB)x$ for every $x \in H$. So we conclude that $\overline{f_B}(B)\overline{f_A}(A) = \xi\overline{f_{AB}}(AB)$. Taking the adjoint, we have $f_A(A)f_B(B) = \overline{\xi}f_{AB}(AB)$. Thus $f_A(A)f_B(B) \approx f_{AB}(AB)$. This showed that the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition. □

Theorem 3.1 presents a general method for constructing sequential products on $\mathcal{E}(H)$. Now, we give two examples.

Example 3.1. Let g be a bounded complex Borel function on [0, 1] such that

$$|g(t)| = \sqrt{t} \text{ for each } t \in [0, 1],$$
$$g(t_1t_2) = g(t_1)g(t_2) \text{ for any } t_1, t_2 \in [0, 1].$$

For each $A \in \mathcal{E}(H)$, let $f_A = g|_{sp(A)}$. Then it is easy to know that $\{f_A\}$ satisfies the sequential product condition. So by theorem 3.1, $A \diamond B = f_A(A)B\overline{f_A}(A) = g(A)B\overline{g}(A)$ defines a sequential product on the standard effect algebra $(\mathcal{E}(H), 0, I, \oplus)$.

It is clear that example 3.1 generalizes Liu and Wu's result in [7].

Example 3.2. Let H be a two-dimensional complex Hilbert space, $\Gamma = \{\gamma \mid \gamma$ be a decomposition of I into two rank-1 orthogonal projections$\}$. For each $\gamma \in \Gamma$, we can represent γ by a pair of rank-1 orthogonal projections (E_1, E_2), if $A \in \mathcal{E}(H)$, $A \notin \text{span}\{I\} = \{zI : z \in \mathbf{C}\}$ and $A = \sum_{k=1}^{2}\lambda_k E_k$, then we say that *$A$ can be diagonalized by γ*.

For each $\gamma \in \Gamma$, we take a $\xi(\gamma) \in \mathbf{R}$. If $A \in \mathcal{E}(H)$, $A \notin \text{span}\{I\}$ and A can be diagonalized by γ, let $f_A(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(A)$.

If $A \in \mathcal{E}(H)$ and $A = \lambda I$, let $f_A(t) = \sqrt{t}$ for $t \in sp(A)$.

Then the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition (see the proof below). So by theorem 3.1, $A \diamond B = f_A(A)B\overline{f_A}(A)$ defines a sequential product on the standard effect algebra $(\mathcal{E}(H), 0, I, \oplus)$.

Proof. Obviously, $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies sequential product condition (i).

Now we show that $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies sequential product condition (ii). Let $A, B \in \mathcal{E}(H)$, $AB = BA$.

(1) If $A = \sum_{k=1}^{2}\lambda_k E_k$, $B = \sum_{k=1}^{2}\mu_k E_k$, $\lambda_1 \neq \lambda_2$, $\mu_1 \neq \mu_2$, let $\gamma = (E_1, E_2)$, we have $f_A(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(A)$, $f_B(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(B)$. So $f_A(A) = A^{\frac{1}{2}+\xi(\gamma)i} = \sum_{k=1}^{2}\lambda_k^{\frac{1}{2}+\xi(\gamma)i}E_k$, $f_B(B) = B^{\frac{1}{2}+\xi(\gamma)i} = \sum_{k=1}^{2}\mu_k^{\frac{1}{2}+\xi(\gamma)i}E_k$.

(1a) If $\lambda_1\mu_1 = \lambda_2\mu_2$, then $AB = \lambda_1\mu_1 I$, so $f_{AB}(t) = t^{\frac{1}{2}}$ for $t \in sp(AB)$, thus we have $f_{AB}(AB) = (AB)^{\frac{1}{2}} = \sqrt{\lambda_1\mu_1}I$, $f_A(A)f_B(B) = \sum_{k=1}^{2}(\lambda_k\mu_k)^{\frac{1}{2}+\xi(\gamma)i}E_k = (\lambda_1\mu_1)^{\frac{1}{2}+\xi(\gamma)i}I = (\lambda_1\mu_1)^{\xi(\gamma)i}f_{AB}(AB) \approx f_{AB}(AB)$.

(1b) If $\lambda_1\mu_1 \neq \lambda_2\mu_2$, then $AB = \sum_{k=1}^{2}\lambda_k\mu_k E_k$, so $f_{AB}(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(AB)$, $f_{AB}(AB) = (AB)^{\frac{1}{2}+\xi(\gamma)i} = \sum_{k=1}^{2}(\lambda_k\mu_k)^{\frac{1}{2}+\xi(\gamma)i}E_k$, thus we have $f_A(A)f_B(B) = \sum_{k=1}^{2}(\lambda_k\mu_k)^{\frac{1}{2}+\xi(\gamma)i}E_k = f_{AB}(AB)$.

(2) If $A = \lambda I$, $B = \sum_{k=1}^{2}\mu_k E_k$, $\mu_1 \neq \mu_2$, let $\gamma = (E_1, E_2)$. Then we have $f_A(t) = t^{\frac{1}{2}}$ for $t \in sp(A)$, $f_B(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(B)$. So $f_A(A) = A^{\frac{1}{2}} = \sqrt{\lambda}I$, $f_B(B) = B^{\frac{1}{2}+\xi(\gamma)i} = \sum_{k=1}^{2}\mu_k^{\frac{1}{2}+\xi(\gamma)i}E_k$, $AB = \sum_{k=1}^{2}\lambda\mu_k E_k$.

(2a) If $\lambda = 0$, then $AB = 0$, $f_{AB}(t) = t^{\frac{1}{2}}$ for $t \in sp(AB)$, so $f_{AB}(AB) = (AB)^{\frac{1}{2}} = 0$. Thus $f_A(A)f_B(B) = 0 = f_{AB}(AB)$.

(2b) If $\lambda \neq 0$, then $f_{AB}(t) = t^{\frac{1}{2}+\xi(\gamma)i}$ for $t \in sp(AB)$. So $f_{AB}(AB) = (AB)^{\frac{1}{2}+\xi(\gamma)i} = \lambda^{\frac{1}{2}+\xi(\gamma)i}\sum_{k=1}^{2}(\mu_k)^{\frac{1}{2}+\xi(\gamma)i}E_k$. Thus $f_A(A)f_B(B) = \sqrt{\lambda}\sum_{k=1}^{2}(\mu_k)^{\frac{1}{2}+\xi(\gamma)i}E_k \approx f_{AB}(AB)$.

(3) If $A = \lambda I$, $B = \mu I$, then $f_A(t) = t^{\frac{1}{2}}$ for $t \in sp(A)$, $f_B(t) = t^{\frac{1}{2}}$ for $t \in sp(B)$. So $f_A(A) = A^{\frac{1}{2}} = \sqrt{\lambda}I$, $f_B(B) = B^{\frac{1}{2}} = \sqrt{\mu}I$. $AB = \lambda\mu I$, $f_{AB}(t) = t^{\frac{1}{2}}$ for $t \in sp(AB)$, $f_{AB}(AB) = (AB)^{\frac{1}{2}} = \sqrt{\lambda\mu}I$. Thus $f_A(A)f_B(B) = f_{AB}(AB)$.

It follows from (1)–(3) that the set $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies sequential product condition (ii). □

4. Properties of the sequential product $\diamond$ on $(\mathcal{E}(H), 0, I, \oplus)$

Now, we study some elementary properties of the sequential product $\diamond$ defined in section 3.

In this section, unless specified, we follow the notations in section 3. We always suppose that $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition. So by theorem 3.1, $\diamond$ is a sequential product on the standard effect algebra $(\varepsilon(H), 0, I, \oplus)$.

Lemma 4.1. *If $C \in \mathcal{E}(H)$, $0 \leqslant t \leqslant 1$, then $f_{tC}(tC) \approx f_{tI}(t)f_C(C)$.*

Proof. Since $\{f_A\}_{A\in\mathcal{E}(H)}$ satisfies the sequential product condition, $f_{tC}(tC) \approx f_{tI}(tI)f_C(C) = f_{tI}(t)f_C(C)$. □

Lemma 4.2. *Let $A \in \mathcal{E}(H)$, $x \in H$, $\|x\| = 1$, $\|f_A(A)x\| \neq 0$, $y = \frac{f_A(A)x}{\|f_A(A)x\|}$. Then $A \diamond P_x = \|f_A(A)x\|^2 P_y$.*

Proof. For each $z \in H$, $(A \diamond P_x)z = f_A(A)P_x\overline{f_A}(A)z = \langle\overline{f_A}(A)z, x\rangle f_A(A)x = \langle z, f_A(A)x\rangle f_A(A)x = \|f_A(A)x\|^2 P_y z$. So $A \diamond P_x = \|f_A(A)x\|^2 P_y$. □

Lemma 4.3. *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, P be a minimal projection in M, $A \in M$, $x \in \mathrm{Ran}(P)$, $\|x\| = 1$. Then $PAP = \omega_x(A)P$, where $\omega_x(A) = \langle Ax, x\rangle$.*

Proof. Since P is a minimal projection in M, by [15, proposition 6.4.3], $PAP = \lambda P$ for some complex number λ. Thus $\langle PAPx, x\rangle = \langle\lambda Px, x\rangle$. So $\lambda = \omega_x(A)$. □

Theorem 4.1. *Let $A, B \in \mathcal{E}(H)$. Then the following conditions are all equivalent:*

(1) $AB = BA$;
(2) $A \diamond B = B \diamond A$;
(3) $A \diamond (B \diamond C) = (A \diamond B) \diamond C$ for every $C \in \mathcal{E}(H)$.

Proof. (1)⇒(2): by theorem 2.2.

(2)⇒(1): by lemma 3.3.

(1)⇒(3): by lemma 3.5.

(3)⇒(1): let $x \in H$, $\|x\| = 1$, $C = P_x$. Then for each $y \in H$,

$$\begin{aligned}\langle f_A(A)f_B(B)P_x\overline{f_B}(B)\overline{f_A}(A)y, y\rangle &= \langle(A \diamond (B \diamond P_x))y, y\rangle\\ &= \langle((A \diamond B) \diamond P_x)y, y\rangle\\ &= \langle f_{A\diamond B}(A \diamond B)P_x\overline{f_{A\diamond B}}(A \diamond B)y, y\rangle.\end{aligned}$$

Since

$$\langle f_A(A)f_B(B)P_x\overline{f_B}(B)\overline{f_A}(A)y, y\rangle = |\langle\overline{f_B}(B)\overline{f_A}(A)y, x\rangle|^2,$$
$$\langle f_{A\diamond B}(A \diamond B)P_x\overline{f_{A\diamond B}}(A \diamond B)y, y\rangle = |\langle\overline{f_{A\diamond B}}(A \diamond B)y, x\rangle|^2,$$

we have $|\langle\overline{f_B}(B)\overline{f_A}(A)y, x\rangle| = |\langle\overline{f_{A\diamond B}}(A \diamond B)y, x\rangle|$ for every $x, y \in H$.

By lemma 3.6, there exists a complex function g on H such that $|g(x)| = 1$ and $\overline{f_B}(B)\overline{f_A}(A)x = g(x)\overline{f_{A\diamond B}}(A\diamond B)x$ for every $x \in H$.

By lemma 3.8, there exists a constant ξ such that $|\xi| = 1$ and $\overline{f_B}(B)\overline{f_A}(A)x = \xi\overline{f_{A\diamond B}}(A\diamond B)x$ for every $x \in H$.

So we conclude that $\overline{f_B}(B)\overline{f_A}(A) = \xi\overline{f_{A\diamond B}}(A\diamond B)$.

Taking the adjoint, we have $f_A(A)f_B(B) = \overline{\xi} f_{A\diamond B}(A\diamond B)$. Thus $\overline{f_B}(B)Af_B(B) = \overline{f_B}(B)\overline{f_A}(A)f_A(A)f_B(B) = \xi\overline{f_{A\diamond B}}(A\diamond B)\overline{\xi} f_{A\diamond B}(A\diamond B) = A\diamond B$. That is, $A\diamond B = \overline{f_B}(B)Af_B(B)$, so by lemma 3.3, we have $AB = BA$. □

Theorem 4.2. *Let* $A, B \in \mathcal{E}(H)$. *If* $A\diamond B \in \mathcal{P}(H)$, *then* $AB = BA$.

Proof. If $A\diamond B = 0$, then by (SEA3) we have $A\diamond B = B\diamond A$, so by theorem 4.1 we have $AB = BA$.

If $A\diamond B \neq 0$. First, we let $x \in \mathrm{Ran}(A\diamond B)$ and $\|x\| = 1$. Then $f_A(A)B\overline{f_A}(A)x = x$. So $\langle B\overline{f_A}(A)x, \overline{f_A}(A)x\rangle = 1$. By the Schwarz inequality, we conclude that $B\overline{f_A}(A)x = \overline{f_A}(A)x$. Thus $Ax = f_A(A)\overline{f_A}(A)x = f_A(A)B\overline{f_A}(A)x = x$. So $1 \in sp(A)$ and $B\overline{f_A}(A)x = \overline{f_A}(A)x = \overline{f_A}(1)x$.

Next, we let $x \in \mathrm{Ker}(A\diamond B)$ and $\|x\| = 1$. Then $f_A(A)B\overline{f_A}(A)x = 0$. So $\langle B\overline{f_A}(A)x, \overline{f_A}(A)x\rangle = 0$. We conclude that $B\overline{f_A}(A)x = 0$.

Thus, we always have $B\overline{f_A}(A) = \overline{f_A}(1)(A\diamond B)$. That is, $f_A(1)B\overline{f_A}(A) = A\diamond B$.

Taking the adjoint, we have $f_A(1)B\overline{f_A}(A) = \overline{f_A}(1)f_A(A)B$.

By lemma 3.2, we have $\overline{f_A}(1)Bf_A(A) = f_A(1)\overline{f_A}(A)B$. So $f_A(1)\overline{f_A}(A)B$ is self-adjoint. By [15, proposition 3.2.8], we have

$sp(f_A(1)\overline{f_A}(A)B)\backslash\{0\} = sp(f_A(1)B\overline{f_A}(A))\backslash\{0\} = sp(A\diamond B)\backslash\{0\} \subseteq \mathbf{R}^+$.

Thus we conclude that $f_A(1)\overline{f_A}(A)B \geqslant 0$.

Since $(f_A(1)\overline{f_A}(A)B)^2 = (\overline{f_A}(1)Bf_A(A))(f_A(1)\overline{f_A}(A)B) = BAB = (f_A(1)B\overline{f_A}(A))(\overline{f_A}(1)f_A(A)B) = (A\diamond B)^2$, by the uniqueness of positive square root, we have $f_A(1)\overline{f_A}(A)B = A\diamond B$. That is, $f_A(1)\overline{f_A}(A)B = \overline{f_A}(1)Bf_A(A) = f_A(1)B\overline{f_A}(A) = \overline{f_A}(1)f_A(A)B = A\diamond B$. Thus, $BA = f_A(1)B\overline{f_A}(A)\overline{f_A}(1)f_A(A) = f_A(1)\overline{f_A}(A)B\overline{f_A}(1)f_A(A) = f_A(1)\overline{f_A}(A)\overline{f_A}(1)f_A(A)B = AB$. □

Theorem 4.3. *Let* $A, B \in \mathcal{E}(H)$. *Then the following conditions are all equivalent:*

(1) $A\diamond(C\diamond B) = (A\diamond C)\diamond B$ *for every* $C \in \mathcal{E}(H)$;
(2) $C\diamond(A\diamond B) = (C\diamond A)\diamond B$ *for every* $C \in \mathcal{E}(H)$;
(3) $\langle(A\diamond B)x, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$ *for every* $x \in H$ *with* $\|x\| = 1$;
(4) $A = tI$ *or* $B = tI$ *for some* $0 \leqslant t \leqslant 1$.

Proof. By theorem 2.4, we conclude that (2)⟺(3)⟺(4).

(4)⇒(1) follows from lemma 2.4 and theorem 2.2 easily.

(1)⇒(4): if (1) holds, then $A\diamond(P_x\diamond B) = (A\diamond P_x)\diamond B$ for each $x \in H$ with $\|x\| = 1$. Without loss of generality, we suppose that $\|f_A(A)x\| \neq 0$. Let $y = \frac{f_A(A)x}{\|f_A(A)x\|}$.

By lemma 4.2 and theorem 2.1,

$$\begin{aligned} A\diamond(P_x\diamond B) &= f_A(A)(P_xBP_x)\overline{f_A}(A) \\ &= f_A(A)(\langle Bx, x\rangle P_x)\overline{f_A}(A) \\ &= \langle Bx, x\rangle(A\diamond P_x) \\ &= \|f_A(A)x\|^2\langle Bx, x\rangle P_y. \end{aligned}$$

By lemma 4.1 and lemma 4.2,

$$\begin{aligned}(A \diamond P_x) \diamond B &= (\|f_A(A)x\|^2 P_y) \diamond B \\ &= f_{\|f_A(A)x\|^2 P_y}(\|f_A(A)x\|^2 P_y) B \overline{f_{\|f_A(A)x\|^2 P_y}}(\|f_A(A)x\|^2 P_y) \\ &= f_{\|f_A(A)x\|^2 I}(\|f_A(A)x\|^2) f_{P_y}(P_y) B \overline{f_{\|f_A(A)x\|^2 I}}(\|f_A(A)x\|^2) \overline{f_{P_y}}(P_y) \\ &= \|f_A(A)x\|^2 P_y B P_y \\ &= \|f_A(A)x\|^2 \langle By, y\rangle P_y.\end{aligned}$$

Thus $\langle Bx, x\rangle = \langle By, y\rangle$. So, we have $\langle \overline{f_A}(A) B f_A(A)x, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$. By lemma 2.5, we conclude that (4) holds. □

Theorem 4.4. *Let $A \in \mathcal{E}(H)$, $E \in \mathcal{P}(H)$. Then the following conditions are all equivalent:*

(1) $A \diamond E \leqslant E$;
(2) $E\overline{f_A}(A)(I - E) = 0$.

Proof. Since $E \in \mathcal{P}(H)$ and $\|\overline{f_A}(A)\| \leqslant 1$, we have

$$\begin{aligned}A \diamond E \leqslant E &\Longleftrightarrow \langle f_A(A) E \overline{f_A}(A)x, x\rangle \leqslant \langle Ex, x\rangle \text{ for every } x \in H \\ &\Longleftrightarrow \|E\overline{f_A}(A)x\| \leqslant \|Ex\| \text{ for every } x \in H \\ &\Longleftrightarrow \overline{f_A}(A)|_{\mathrm{Ker}(E)} \subseteq \mathrm{Ker}(E) \\ &\Longleftrightarrow E\overline{f_A}(A)(I - E) = 0.\end{aligned}$$

□

Corollary 4.1 [14]. *Let $A \in \mathcal{E}(H)$, $E \in \mathcal{P}(H)$. Then the following conditions are all equivalent:*

(1) $A^{\frac{1}{2}} E A^{\frac{1}{2}} \leqslant E$;
(2) $AE = EA$.

Proof. (2)⇒(1) is trivial.

(1)⇒(2): let $f_B(t) = \sqrt{t}$ for each $B \in \mathcal{E}(H)$ and $t \in sp(B)$, then $\{f_B\}_{B\in\mathcal{E}(H)}$ satisfies the sequential product condition. For this sequential product, $A \diamond E = A^{\frac{1}{2}} E A^{\frac{1}{2}}$. So by theorem 4.4, we have $EA^{\frac{1}{2}}(I - E) = 0$. That is, $EA^{\frac{1}{2}} = EA^{\frac{1}{2}}E$. Taking the adjoint, we have $EA^{\frac{1}{2}} = A^{\frac{1}{2}}E$. Thus $AE = EA$. □

Corollary 4.2. *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, $\mathcal{E}(M) = \{A \in M | 0 \leqslant A \leqslant I\}$, P or $I - P$ be a minimal projection in M. Then for every $A \in \mathcal{E}(M)$, the following conditions are all equivalent:*

(1) $A \diamond P \leqslant P$;
(2) $AP = PA$.

Proof. (2)⇒(1): by theorem 2.2, $A \diamond P = AP = PAP \leqslant P$.

(1)⇒(2): if P is a minimal projection in M, then by theorem 4.4 we have $P\overline{f_A}(A)(I - P) = 0$, that is, $P\overline{f_A}(A) = P\overline{f_A}(A)P$.

Let $x \in \mathrm{Ran}(P)$ with $\|x\| = 1$. Then by lemma 4.3 we have $P\overline{f_A}(A)P = \omega_x(\overline{f_A}(A))P$. So $P\overline{f_A}(A) = \omega_x(\overline{f_A}(A))P$. Taking the adjoint, we have $f_A(A)P = \omega_x(f_A(A))P$. By lemma 3.2, we have $Pf_A(A) = \overline{\omega_x(\overline{f_A}(A))}P = \omega_x(f_A(A))P$. Thus $Pf_A(A) = f_A(A)P$. Taking the adjoint, we have $P\overline{f_A}(A) = \overline{f_A}(A)P$. So, $PA = Pf_A(A)\overline{f_A}(A) = f_A(A)P\overline{f_A}(A) = f_A(A)\overline{f_A}(A)P = AP$.

If $I - P$ is a minimal projection in M. By theorem 4.4 we have $P\overline{f_A}(A)(I - P) = 0$. Taking the adjoint, we have $(I - P)f_A(A)P = 0$. That is, $(I - P)f_A(A) = (I - P)f_A(A)(I - P)$. Similar to the proof above, we conclude that $(I - P)A = A(I - P)$. So $AP = PA$. □

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